**Beenaker Derivation of DMPK**

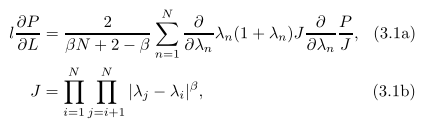
It seems that weak scattering/disorder is considered to be ℓ > λF. And strong disorder is ℓ < λF I guess. This scale would accord with the localization lengths derived by the path integral approach.

In 1D, conductance always scales to 0 with length g ~ exp(-L/ξ), and we can estimate the localization length as the length where weak localization corrections cut-off g. So ξ ~ ℓ. He says that this was proven for the case of weak disorder, by some people calculating the transmission coefficient in a weak scattering 1D chain. And he says the entire probability distribution was also found. He says that other people tackled the strong disorder limit, but doesn’t say what those results were. What would change? ξ?

In 2D he says that g → 0 as function of L/ξ, where ξ is estimated same way ξ ~ exp(ℓ), but he says the decay is not necessarily exponential.

And then in 3D he says that this depends on disorder. If ℓ > ℓC, then it will scale as g ~ L, and if ℓ < ℓc, then it will scale exponentially g ~ exp(-L/ξ), where ξ ~ 1/(ℓC – ℓ)ν.

Now he focuses on Q1D. The conductance can be calculated from the Landauer formula g = ΣTn. He says that there is an effective eigenvalue repulsion so that (because of this interaction) we now have a different scaling behavior and governing PDE. The localization length increases as result of repulsion and now we have ξ ~ Nℓ. Because this length scale is larger than ℓ, we can distinguish three regimes: L < ξ, L ~ ξ, and L > ξ. For the first regime, he says that g(L/ξ) ~ ξ/L, but when L > ξ it goes as exp(-L/ξ). The governing PDE is:



β is parameter that governs the symmetry of H. Observe that when N = 1, β doesn’t matter (cancels out). I guess this is because there are no closed loop paths possible and so B field cannot affect the phase? Same with spin-orbit scattering?

He says that to derive this PDE, one must assume a weak scattering limit, namely that ℓ > λF. This is so that scattering in the thin slice may be treated perturbatively. Also the isotropy assumption – that the flux incident in one channel is evenly distributed though out the other channels – requires the assumption that L >> W (width) since we need many scattering events. Even if L = W, he says that as long as W << ξ then we may still use the DMPK equation. He mentions an alternative of ‘equivalent scattering channels’ given by Mello and Tomsovic (1991, 1990) (pg. 32). Also see Dhorokov (1988) pg. 32 who constructed a model where the equivalent chain assumption holds exactly.

When adding a perturbative slice of thickness δL, he says that we require δL << ℓ so that the scattering can be treated perturbatively. Note that scattering can still happen in the slice because ℓ is just the mean free path, so there will be some scatterers within the slice. The perturbative series for δTn does come out in powers of δL/ℓ so this seems to be required. Look into this a bit. Also require assumption that δL (thickness of slice) >> λF so that the propagation of evanescent modes may be ignored. Have to check out the details of this, mentioned in Mello’s book. Maybe the evanescent modes can be ignored? These two inequalities together require that ℓ >> λF which is the weak scattering limit.

Now talking about how to justify definition <λ´n> ~ δL/ℓ.

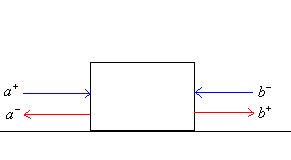
He talks about how ξ changes depending on B, and spin-orbit effects. Basically ξ = βNℓ. He further mentions that work has been done on these localization lengths for 2D and 3D and the results are apparently not universal, as they are for Q1D.

**P(g) scaling equation set up**

Let’s examine the Landauer formula for g.



Our goal will be to determine the probability distribution function for the eigenvalues of |tt†|, Ti, as a function of length. We will use length as the parameter since we know that all Ti ought to be 1 at 0 length. And from this distribution we can then construct the probability distribution of conductances. So to begin let’s examine the typical set up a sample connected by leads to two reservoirs maintained at their respective chemical potentials. On each side we’ll have incoming and outgoing electrons propagating in the N available channels. Channels come from transverse boundary conditions and longitudinal periodicity in the leads I suppose.



So the wave function describing the currents in the leads would look like,

 on the left hand side and

 on the right hand side

(As a note for later, if SRS isn’t present then we should include the spin eigenkets too). In principle, one would have to start with a well defined ψL, and then solve Schrödinger’s equation in the sample, and apply continuity, differentiability conditions at the interfaces, to obtain ψR. I imagine that you would have to multiply both sides by Φm(x,y) and integrate both sides - leaving you with an equation for just the a’s and b’s. But instead of using up our d.o.f. by defining ψL unambiguously, we use the incomming channels of ψL and ψR rather, by demanding that the currents in each channel be normalized to unity (kind of like is done in the tunneling problems in 1D).

**Previous results formulated in terms of transmission eigenvalues, eigenvectors**

So, we have from the Landauer formula that,



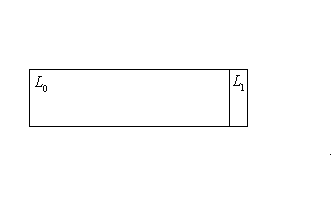
where Tn are the eigenvalues of tt†. In the metallic regime, these eigenvalues must go as, as we found above, Tn ~ ℓ/Lz. In the insulating regime, the conductance goes as exp(-L/ξ), and so we would expect that transmission eigenvalues would scale like this as well. I would expect them to be scale independent in the critical regime.

What do the eigenvectors of tt† signify? Well, when we’re in the metallic regime, it is said that the υ (roughly the matrix of eigenvectors) is isotropically distributed. This would mean that the eigenvectors correspond to N dimensional phasors? This is indicative of the ability of any particular channel to tunnel into any other. This would be expected because in the diffussive regime, if we send in an e- with momentum k, by the time it arrives at the other end of the metal, it will have scattered so many times, that its final momentum, k´, will have an equal probability of being anywhere on the Fermi surface. When the eigenvectors are of the form (0, …,0,1,0,…,0) in position space. Then this is indicative of an insulating state, where many of the channels are closed off. So this would mean that a particle in channel k cannot tunnel into channel ℓ. Closing off of channels corresponds to mobility edge moving inward I suppose. This doesn’t happen in Q1D because ξ is always >> the transverse length L and so the particles can always diffuse in the transverse direction, and can hence end up with whatever k´ given initial k. The 3D situation is illustrated below; the LHS represents weak disorder and the RHS strong disorder. In the strong disorder case, there is at best a single path which will allow the e- to tunnel out of the sample.



**P(g) scaling equation in Q1D: The DMPK equation**

Note the introduction to this section in Beenaker’s review - it contains some interesting information on scaling in general. This formalism should be applicable to light scattering from random dielectric materials as well. In which case, we would obtain the probability distribution of transmittance, reflection.



The basic idea is to use a correspondence between the Fokker-Plank equation and our scaling equation. In the former, we examine the evolution of a coordinate **x**(t) as a function of time, subject to a driving force, and a random internal force. Here we are examining the evolution of transmission matrix (tt†) eigenvalues **T**(L) as a function of length. We want to evaluate the probability distribution of **x**(t) in the former, and so we attempt to evaluate P(**x**,t) = <δ(**x** – **x**(t))>, where < > denotes an average over ensembles. And here we wish to do similarly. We wish to evaluate the probability distribution of eigenvalues P(**T**,L) = <δ(**T** – **T**(L))>, where < > denotes an average over ensemble with the same macroscopic disorder. So in the former we develop a differential equation for P(**x**,t) by considering, deterministically for a given internal force, how **x**(t) changes as we add a differential time Δt to it; we examine **x**(t + Δt). And here we examine **T**( L + δL). We compute the former via the Langevin equation – or whatever determistic equation we have. And in the latter we use perturbation theory to evaluate the change in eigenvalues. Then we perform the average over ensembles in both cases. And we observe in the former case that it is the first two moments themselves which are sufficient to determine P(**x**,t) and similarly we’ll have for P(**T**,L) since higher moments go as δL1+. So then we are done. We have,



and so have,



So basically, we just need the first two moments of eigenvalues (provided higher moments don’t go higher than δL), and we’re set.

**Determining expression for  in order to get expression for T(L + δL)**

We now use the composition property of the M’s to determine how tt+ will change. Consider a wire of length L0 to which we attach a segment L1. The combined system has a length . The total transmission matrix, as aforementioned would be



(remember, multiply by M0 to get to the right, and then by M1 to go further). Beenaker says that this requires that we ignore evanescent modes, which is justified if L1 >> λF. One could say that we require that the particle can be considered semi-classically propagating through the sample? If so, then,



Multiplying them all together, we get t2 in terms of the other 1 and 0 elements, presumably after using some of the other relationships.



Now want to calculate the probability distribution of the eigenvalues of  . So we form this expression using the above and come to



grouping all the extra terms into ω. We want to calculate the eigenvalues of |t|2.  Supposing that we know the eigenvalues of |t0|2, we will use standard perturbation theory on the ω term. We’ll denote by Tn the transmission eigenvalues of the |t0|2 matrix, and Tn + δTn the transmission eigenvalues of the |t2|2 matrix. Then to second order we have



where we evaluate ω in the eigenbasis.

**Putting in terms of polar matrices so can perform the disorder average**

Now we would like to use the polar decomposition of M, to write the t’s, r’s, etc. in terms of the polar matrices: U0, U0´, V0, V0´, T0; and U1, U1´,V1, V1´, T1. We could use,



(which we found above) perhaps, but he uses a different decomposition.



Well, these seem to be at odds, but I’ll just go on. He carries out the calculation and obtains,



defining  and . Now in order to make further progress we have to know how to disorder average.

**Disorder average over M1**

So we take an ensemble (disorder) average over L­1 polar matrices. The ‘equation of motion’ of the T0n­’s involve both the initial U0, V0, and T0, as well as the incremental U1, V1, T1. (Note T2 = T0 + δT is not T0 + T1 – otherwise transmission would increase as keep adding layers). The disorder averages over the M1’s usually assume weak scattering, etc. The conditions are:



This condition implements the constraint M1 → I in the small δL limit.



I think this is equivalent to the following disorder average over T1 below. No assumptions are made with respect to the distribution of the  matrix, except that its first moment defines the mean free path via:



Note that this introduces a requirement that L1 << ℓ in order for the perturbation theory to work. Taken with the earlier condition that L1 >> λF­, this requires that λF << ℓ, which means that we must have only weak disorder. But does this then rule out the possibility of a scaling equation for the insulating state of a metal, where λF ~ ℓ, or greater.

And we define , and hence. I’m guessing that we derive this claim from the weak localization correction to the conductivity in 1D (could say, for a single channel): . And next, higher moments are given by:



All of it basically seems to say that the transmission eigenvalues are reduced proportionally to first order, but no mixing of channel indices occurs.

**Disorder average over M0**

The disorder averages over the M0’s (excluding the T0’s as they are what we’re developing an equation for) assume isotropy in the Q1D case. So we assume a uniform distribution of unitary matrices – the Q1D assumption. Note that we’re assuming that U and V are uniformly distributed. I would’ve assumed that U0 and V0 are, but that U1 and V1 are sort of close to being I? If this is the case, then U and V are uniformly distributed too.

And then once those disorder averages have been used, we may use the Q1D isotropy assumption on U0 and V0. In any event, an ensemble average of a function of the unitary matrices is defined as



where the differential is an invariant measure, and U and V are assumed to be independent, as they are when. To first order in (1/N) we can think of this as just an integral over independent random variables (recall U, V matrix elements have real and imaginary parts), with a gaussian distribution about 0, and variance 1/2N, so that



When you go to order the exact expression deviates somewhat from this. If you change to 1(4), (TR symmetry with/without spin-rotation symmetry), then the average changes a bit because the polar matrices are then related to each other because of the extra symmetries that S possesses. For instance the case would look like

 because 

**Disorder averaged first two moments**

Now we use these formulas to calculate to first order in . His expressions are...



The third and higher moments vanish to order as we can see.

**Use Fokker-Plank analysis to obtain result**

And appealing to the Fokker-Planck equation analysis, We have that  follows



We come to the DMPK equation upon change of variables 



Note that , of later fame. The derivation here rested on the isotropy of the scattering matrix assumption. Note that since βNℓ/2 is the localization length, ξ, we can write this as:



**Distribution in Q1D**

If Lz >> ξ, we will expect insulating behavior. We can solve the DMPK directly in the insulating regime. We use the fact that the eigenvalues are exponentially separated, we can split the DMPK equation into separable pieces, and obtain:



Ordering the eigenvalues from least to greatest we get,





recall that,



and so



The distribution of the first eigenvalue is therefore,



and relating we have,



so,



So now defining Γ = γ/2s, we have,



So this identifies our localization length as lng ~ L/ξ → Γ = ξ/L. And using that γ = βN + O(1) and we have,



And so we have,



A more detailed analysis of the distribution reveals that there is an exp. cutoff at g = 1.

In crossover regime its almost flat (very broad) for g<1, and again cutoff at g = 1.

If ℓ << LZ << ξ, then we can get metallic like behavior, even in Q1D (though not in 1D he says since ℓ = ξ). And we find that,



If we calculate the conductance distribution we find,



and so,



where <g> clearly scales ~ σ/Lz in accordance with the Ohm’s law scaling equation. We also see the weak localization correction tacked on. The variance is shown to be universal, independent of disorder (so long as the inequality above is satisfied), length, etc. A more careful analysis again shows an exponential cutoff around g = 1 (actually at g = 1 + e-3/Γ) which disappears (the magnitude of the discontinuity) smoothly as a function of Γ, namely e-3/Γ as well.